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A SPLINE BASED TECHNIQUE FOR COMPUTING RICCATI OPERATORS AND FEEDBACK CONTROLS IN REGULATOR PROBLEMS FOR DELAY EQUATIONS

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ABSTRACT

We consider the infinite interval regulator problem for systems with delays. A spline approximation method for computation of the gain operators in feedback controls is proposed and tested numerically. Comparison with a method based on "averaging" approximations is made.

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1. INTRODUCTION

The problem of constructing feedback controls for hereditary or delay systems is not new and there is a rather extensive literature pertaining to several aspects of this problem. We refer the reader to the surveys of Ross [24], Alekal, et.al. [1], and section 5 of Banks and Burns [5] for accounts of some of the previous pertinent results. Among the fundamental earlier contributions are those of Krasovskii [17], [18] (establishment of the functional form of optimal feedback for delay systems and early use of an "averaging" type approximation scheme), Eller, et.al. [13] and Ross [24], [25] (derivation of Riccati type equations for the feedback gains in the functionals and methods for computing these gains), and Delfour [12] (convergence analysis of an "averaging" scheme for approximate solution of an operator form of the Riccati type equations for the feedback gains). More recently, Gibson [14] and Kunisch [19] have made important contributions which we shall discuss in the context of our presentation below.

Our own renewed interest in feedback controls for delayed systems was motivated by problems arising in the design of controllers for a liquid nitrogen wind tunnel (the National Transonic Facility or NTF) currently under construction by NASA at its Langley Research Center in Hampton, Va. With this wind tunnel it is expected that researchers will be able to achieve an order of magnitude increase in the Reynolds number over that in existing tunnels while maintaining reasonable levels of dynamic pressure. Test chamber temperatures (the Reynolds number is roughly inversely proportional to temperature) will be maintained at cryogenic levels by injection of liquid nitrogen as a coolant into the airstream near the fan section of the tunnel. In addition to a gaseous nitrogen vent to help control pressure, motor driven fans will be used as the primary regulator of Mach number. Fine control of Mach number will be effected through changes in inlet guide vanes in the fan section. Schematically, the tunnel can be depicted as in Figure 1.1

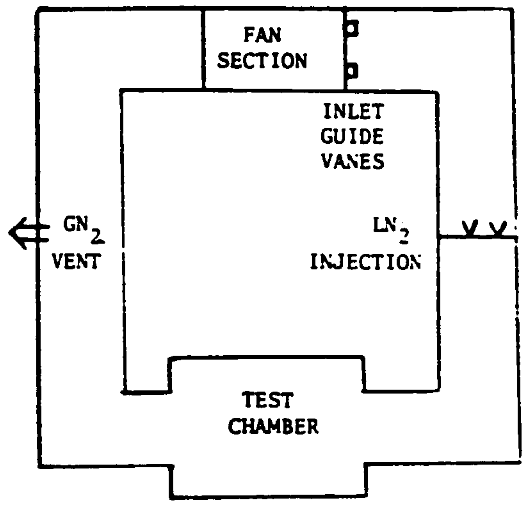

FIGURE 1.1

The basic physical model relating states such as Reynolds number, pressure, and Mach number to controls such as LN_2 input, GN_2 bleed, and fan operation involves a formidable set of partial differential equations (the Navier-Stokes theory) to describe fluid flow in the tunnel and test chamber. This model has, not surprisingly, proved to be very unwieldy from a computational viewpoint and is difficult, if not impossible, to use directly in the design of sophisticated control laws. (Both open loop and feedback controllers are needed for efficient operation of the tunnel - and this is a desirable goal since cost estimates for liquid nitrogen alone are \$6.5 × 10^6 per year of operation.) In addition to the design of both open loop and closed loop controllers, parameter estimation techniques will be useful once data from the completed tunnel is available (current investigations involve use of data from a 1/3 meter scale model of the tunnel).

In view of the schematic in Figure 1.1, it is not surprising that engineers (e.g., see [3] and [15]) have proposed design of control laws for subsystems modeled by lumped parameter models (the variables represent values of states and controllers at various discrete locations in the tunnel and test chamber) with transport delays to account for flow times in sections of the tunnel. A specific example is the model [3] for the Mach no. control loop in which variations in the Mach no. (in the test chamber) are, to first order, controlled by variations in the inlet guide vane angle setting (in the fan section) - i.e., $\delta M(t) \sim \delta\theta(t-r)$ where r represents a transport time from the fan section to the test chamber. More precisely, the proposed equation relating the variation δM (from steady state operating values) in Mach no. to the variation $\delta\theta$ in guide vane angle is

$$\tau \delta\dot{M}(t) + \delta M(t) = k\delta\theta(t-r)$$

while the equation relating the guide vane angle variation to that $\delta\theta_A$ of an actuator is

$$\delta\ddot{\theta}(t) + 2\zeta\omega\delta\dot{\theta}(t) + \omega^2\delta\theta(t) = \omega^2\delta\theta_A(t).$$

Rewriting the system in vector notation, one thus finds that the Mach no. control loop involves a regulator problem for the equation

$$(1.1) \quad \dot{x}(t) = A_0x(t) + A_1x(t-r) + B_0u(t)$$

where $x = (\delta M, \delta\theta, \delta\dot{\theta})$, $u = \delta\theta_A$. Here the control is the guide vane angle actuator input. A similar 4-vector system problem can be formulated in the case where one treats the actuator rate $\delta\dot{\theta}_A$ as the control - see [3], [11]. We shall return to examples such as (1.1) for the NTF in section 4 below where we present numerical results obtained using the methods we propose.

Several recent contributions to the literature on numerical methods for delay systems prompt the techniques we present in this paper. A rather complete convergence analysis (along with numerical results) of the so-called "averaging" scheme applied to open loop control problems for delay systems was given in Banks and Burns [5]. The analysis was based on approximation results for linear semigroups involving the Trotter-Kato theorem (a functional analytic version of the Lax Equivalence theorem: consistency plus stability implies convergence). Gibson [14] and Kunisch [19] have shown that these same tools can be used to develop a convergence theory for approximations of the feedback gains based on the "averaging" techniques. Subsequent to the development of "averaging" methods for delay systems (which result in essentially first order numerical schemes), Banks and Kappel [9] developed higher order approximation schemes based on spline approximations. In numerous situations ([4], [6], [7], [8], [9]) these methods have proven superior computationally to the popular "averaging" techniques. In this paper we show how one can use spline based computational schemes to obtain the gains in the feedback controllers for delay systems. We present a summary of our numerical findings with these methods which support the efficacy of the proposed schemes.

Our presentation is as follows: In section 2 we summarize those facts from the literature on delay systems needed to discuss and develop our approximation techniques. Section 3 is then devoted to a careful explanation of the proposed schemes, hopefully in sufficient detail to permit readers to develop their own computational packages should they so desire. We report on our numerical experience with the spline based schemes in section 4 where we also compare our findings to those obtained using the "averaging" methods. Finally we discuss briefly in section 5 some of the theoretical aspects of the spline techniques.

The notation we use throughout is rather standard with the following exception. We shall be dealing with vector systems but shall not always make this precise when no loss of understanding results. For example, if x is an n-vector valued function with components in the Sobolev space H^1, we shall simply write $x \in H^1$. We shall only use transpose notation where it is essential; e.g., if Q_0 is an $n \times n$ matrix we shall write xQ_0x instead of the more conventional x^TQ_0x.

2. FEEDBACK CONTROLS FOR DELAY SYSTEM PROBLEMS

In light of the motivation above, we consider the control problem of finding an m-vector valued L_2 control $\bar{u}$ which minimizes

$$(2.1)\qquad J(u;\, \eta,\psi) = \int_0^\infty [x(t)Q_0x(t) + u(t)Ru(t)]dt$$

subject to the n-vector system

$$(2.2)\qquad \dot{x}(t) = Lx_t + B_0u(t)\ , \qquad t \geq 0\ ,$$

$$(2.3)\qquad x(0) = \eta,\quad x_0 = \psi,$$

where Q_0, R are symmetric $n \times n$ and $m \times m$ matrices, respectively, with $Q_0 \geq 0$, $R > 0$, B_0 is an $n \times m$ matrix, and ψ is an n-vector function with components in $L_2(-r,0)$ - (we denote this by $\psi \in L_2^n(-r,0)$). Following standard notation, the symbol x_t denotes the function $\theta \to x(t+\theta)$, $-r \leq \theta \leq 0$, and we assume the linear operator L has the form

$$L\phi = \sum_{i=0}^{\nu} A_i\phi(-r_i) + \int_{-r}^{0} D(\theta)\phi(\theta)d\theta$$

where $0 = r_0 < r_1 < \ldots < r_\nu = r$, A_i, $i = 0, 1, \ldots, \nu$, are $n \times n$ matrices, and D is an $n \times n$ matrix function with components in $L_2(-r,0)$. This operator and the system (2.2), (2.3) can be given a proper interpretation for initial data ψ and controls u in L_2 and, indeed, one can establish existence of a unique solution $x \in H^1$ on any finite interval [0,T] where the equation (2.2) is satisfied in the usual Caratheodory sense (i.e., almost all t) - see [9].

Assuming for the moment that a solution to the above control problem exists in closed loop form, it can be shown (see [17], [14]) to have the form

$$(2.4)\qquad \bar{u}(t) = -\,[K_0x(t) + \int_{-r}^{0} K_1(\theta)x(t+\theta)d\theta]$$

where the $m \times n$ gain matrices satisfy certain Riccati-like systems of equations ([13], [24], [1], [14]). Our goal here is to discuss numerical approximations to K_0 and K_1 which, when applied to (2.2), (2.3), (2.4), yield a near optimal performance. It has been understood for some time that we are in this case dealing with feedback controls for a infinite dimensional state system. This system can be succinctly formulated abstractly (e.g., see [5], [9], [12], [14]) in a manner that facilitates convergence analyses for approximation schemes. While we shall not pursue a convergence analysis in this paper, it is convenient in discussing our numerical methods and results to use this formulation and the corresponding notation.

To this end, we let

(2.5) $\quad z(t) = (x(t), x_t)$

where x is the solution of (2.2), (2.3). Define Z to be the product space $R^n \times L_2^n(-r,0)$ with the usual product Hilbert space topology (and inner product) and let $D(A) \equiv \{(\xi,\phi) \in Z\colon \xi = \phi(0),\ \phi \in H^1(-r,0)\}$ be the domain for the linear operator A given by $A(\phi(0),\phi) = (L\phi, \dot{\phi})$. Recalling (2.1) and (2.2), we define the linear operators $Q : Z \to Z$ and $\mathcal{B} : R^m \to Z$ by $Q(\xi,\phi) = (Q_0\xi, 0)$ and $\mathcal{B}v = (B_0 v, 0)$. Then our original optimization problem for (2.1) - (2.3) can be reformulated as the equivalent problem of minimizing

(2.6) $\quad J(u;z_0) = \int_0^\infty \{<Qz(t),z(t)> + u(t)Ru(t)\}dt$

over $u \in L_2$ subject to the evolution equation constraint

(2.7) $\quad \dot{z}(t) = Az(t) + \mathcal{B}u(t), \qquad t \geq 0,$

(2.8) $\quad z(0) = z_0 = (\eta, \psi).$

It is known [5], [9] that A generates a C_0-semigroup $\{S(t)\}$ of solution operators and that z defined by (2.5) is the unique mild solution of (2.7), (2.8). That is, z is given by

$$(2.9) \qquad z(t) = S(t)z_0 + \int_0^t S(t-\sigma)\mathcal{B}u(\sigma)d\sigma.$$

If we define an admissible control for our problem corresponding to the initial condition $z_0 \in Z$ to be an m-vector function u which is integrable on $(0,\infty)$ and for which $J(u;z_0)$ is finite; and if we make the assumption that the operators Q_0 and L are such that any admissible control corresponding to the initial condition $z_0 \in Z$ drives the resulting solution of the state equation (2.7) to zero asymptotically, then we may use results due to Gibson [14] to characterize the solution to the problem in feedback form. More precisely, if there exists an admissible control corresponding to each initial condition $z_0 \in Z$ (or equivalently the system (2.7) is stabilizable, see definition 2.3 and corollary 4.1 of [14]), then there exists a nonnegative, selfadjoint linear operator Π on Z which satisfies the Riccati algebraic equation

$$(2.10) \quad A^*\Pi + \Pi A - \Pi \mathcal{B}R^{-1}\mathcal{B}^*\Pi + Q = 0 .$$

Moreover, under the assumption made above there exists at most one such solution and the unique solution to the problem (2.6) - (2.8) can be given in feedback form by

$$(2.11) \quad \bar{u}(t) = -R^{-1}\mathcal{B}^*\Pi z(t), \qquad 0 < t < \infty,$$

and

$$\min_{v \text{ admissible}} J(v;z_0) = \langle \Pi z_0, z_0 \rangle .$$

The operator Π can be written as a matrix of linear operators

$$\Pi = \begin{bmatrix} \Pi^{00} & \Pi^{01} \\ \Pi^{10} & \Pi^{11} \end{bmatrix}$$

where $\Pi^{00} : R^n \to R^n$ and $\Pi^{11} : L_2^n(-r,0) \to L_2^n(-r,0)$ are nonnegative and selfadjoint, $\Pi^{10} \in L_2^{n\times n}(-r,0)$ and $\Pi^{01} = \Pi^{10*}$ with

$$(2.12) \quad \Pi^{01}\phi = \int_{-r}^{0} \Pi^{10}(\theta)^T \phi(\theta)d\theta, \qquad \phi \in L_2^n(-r,0) .$$

If we recall the definition of the operator $\mathcal{B}$ and assume that the system (2.7) is stabilizable,then under the assumption made above, (2.5), (2.11) and (2.12) yield that the unique solution to the problem for (2.1) - (2.3) is given in feedback form by

$$(2.13) \quad \bar{u}(t) = - R^{-1}B_0^T[\Pi^{00}x(t) + \int_{-r}^{0} \Pi^{10}(\theta)^T x(t+\theta)d\theta]$$

with (for $z_0 = (\eta,\psi)$)

$$\min_{v \text{ admissible}} J(v;\eta,\psi) = \langle \Pi z_0, z_0 \rangle .$$

That is, the gains K_0, K_1 of (2.4) are given by $R^{-1}B_0^T \Pi^{00}$ and $R^{-1}B_0^T(\Pi^{10})^T$, respectively, and can be obtained by solving the Riccati equation (2.10).

3. THE APPROXIMATION SCHEME

In this section we develop and discuss the implementation of a spline based computational scheme which yields a sequence of finite dimensional operators $\{\Pi_N\}$ which approximate Π, the solution to the operator Riccati algebraic equation given by (2.10). The Π_N are found by solving standard matrix Riccati algebraic equations, and are then used to construct feedback controls which approximate (2.11) and which produce near optimal performance by the system (2.7) (2.8) as measured by the functional (2.6).

The approach we take is based largely upon the spline approximation framework developed in [9] for the approximation of solutions of linear functional differential equations. We summarize briefly the essentials of that development. Let Z_N be a sequence of spline based subspaces of Z satisfying $Z_N \subset D(A)$ $N = 1, 2, \ldots$. Let $P_N : Z \to Z_N$ denote the corresponding sequence of orthogonal projections of Z onto Z_N computed with respect to the weighted inner product $\langle \cdot\ \cdot \rangle_g$ on Z given by

$$\langle(\eta,\phi),(\xi,\psi)\rangle_g = \eta^T\xi + \int_{-r}^{0} \phi(\theta)^T\psi(\theta)g(\theta)d\theta$$

where

$$g(\theta) = \begin{cases} 1 & -r \leq \theta < -r_{\nu-1} \\ 2 & -r_{\nu-1} \leq \theta < -r_{\nu-2} \\ \cdot & \\ \cdot & \\ \cdot & \\ \cdot & \\ \nu-1 & -r_2 \leq \theta < -r_1 \\ \nu & -r_1 \leq \theta \leq 0. \end{cases}$$

Define the linear operators A_N and Q_N on Z_N and $\mathcal{B}_N : R^m \to Z_N$ by $A_N = P_N A$, $\mathcal{B}_N = P_N \mathcal{B}$ and $Q_N = P_N Q$, respectively, and let Π_N be a nonnegative selfadjoint solution to the Riccati algebraic equation in Z_N given by

$$(3.1) \qquad A_N^* \Pi_N + \Pi_N A_N - \Pi_N \mathcal{B}_N R^{-1} \mathcal{B}_N^* \Pi_N + Q_N = 0 \ .$$

The existence and uniqueness of solutions of (3.1), which are related to the existence and uniqueness of solutions of (2.10) and certain properties of the approximation scheme itself, will be discussed in section 5. For the present, however, we assume that for all N sufficiently large, a solution Π_N exists with $\Pi_N \geq 0$ and $\Pi_N^* = \Pi_N$.

The use of the weighted inner product $\langle\cdot,\cdot\rangle_g$ in place of the standard inner product on Z in computing the projections P_N (and therefore the operators A_N) insures that the operators A_N satisfy a uniform dissipative inequality of the form

$$\langle A_N z, z\rangle_g \leq \beta \langle z,z\rangle_g \ , \qquad z \in Z_N,$$

and hence that the solutions of the finite dimensional ordinary differential equation initial value problem in Z_N

$$(3.2) \qquad \dot{z}_N(t) = A_N z_N(t) + \mathcal{B}_N u(t), \qquad t \geq 0 \ ,$$

$$z_N(0) = P_N z_0$$

approximate the solution of (2.7),(2.8) (see [9]). It is this fundamental convergence result which forms the theoretical foundation for the schemes being developed here.

Since the domain of the operators P_N is $Z = R^n \times L_2^n(-r,0)$, the operators $\Pi_N P_N$ can be written as the matrix of linear operators given by

$$\Pi_N P_N = \begin{bmatrix} \Pi_N^{00} & \Pi_N^{01} \\ \Pi_N^{10} & \Pi_N^{11} \end{bmatrix}$$

where the $n \times n$ matrix Π_N^{00} and $\Pi_N^{11} : L_2^n(-r,0) \to L_2^n(-r,0)$ are nonnegative and selfadjoint, Π_N^{10} is an $n \times n$ matrix valued function with components in L_2, and $\Pi_N^{01} = \Pi_N^{10*}$ with

$$\Pi_N^{01}\phi = \int_{-r}^{0} \Pi_N^{10}(\theta)^T\phi(\theta)d\theta, \qquad \phi \in L_2^n(-r,0).$$

If the approximating optimal controls in feedback form for the problem involving (2.6) - (2.8) are defined by

(3.3) $\quad \bar{u}_N(t) = -R^{-1}B_N^* \Pi_N P_N z(t)$,

then the approximate solutions to our problem take the form

(3.4) $\quad \bar{u}_N(t) = -[R^{-1}B_0^T\Pi_N^{00}x(t) + \int_{-r}^{0} R^{-1}B_0^T\Pi_N^{10}(\theta)^T x(t+\theta)d\theta]$

with the corresponding approximating optimal trajectories being given by the solutions to

(3.5) $\quad \dot{x}(t) = (A_0 - B_0R^{-1}B_0^T\Pi_N^{00})x(t) + \sum_{j=1}^{\nu} A_j x(t-r_j)$

$$+ \int_{-r}^{0} (D(\theta) - B_0R^{-1}B_0^T\Pi_N^{10}(\theta)^T)x(t+\theta)d\theta$$

for any initial conditions $x(0) = \eta \in R^n$ and $x_0 = \psi \in L_2^n(-r,0)$. In addition, the optimal cost can be approximated by

(3.6) $\langle \Pi_N P_N(\eta,\psi),\ P_N(\eta,\psi)\rangle$.

Equation (3.1) is an operator equation, and thus is not suitable for computational purposes in its present form. In order to find the matrix form of (3.1) a basis for Z_N must be chosen and matrix representations for the operators A_N, A_N^*, $\mathcal{B}_N$, $\mathcal{B}_N^*$ and Q_N with respect to this basis must be computed. The adjoint operators A^* and $\mathcal{B}^*$ (and therefore their approximations A_N^* and $\mathcal{B}_N^*$) may be computed with respect to either the standard inner product on Z, $\langle\cdot,\cdot\rangle$, or the weighted inner product $\langle\cdot,\cdot\rangle_g$. Indeed, the fact that

$$\langle Qz,z\rangle_g = \langle Qz,z\rangle$$

for all $z \in Z$ implies that the abstract regulator problem given by (2.6), (2.7) and (2.8) can be formulated in the space Z using either inner product and still be equivalent to our original control problem. However, the expressions for the matrix representations for the operators A_N^* and $\mathcal{B}_N^*$ are simplified if the $\langle\cdot,\cdot\rangle_g$ inner product is employed (in this case of course it must also be used in (3.6)). When the discrete delay part of L consists of only a single delay term (i.e., $\nu=1$), then $g(\theta) \equiv 1$ and the two inner products are the same.

We shall outline the necessary procedure for finding matrix representations in the case of "linear" or first order spline functions; however the ideas presented are easily extended to the case of cubic or higher order spline functions.

For each N = 1, 2, ..., and each $\theta \in [-r,0]$, let

$$\phi_j^N(\theta) = \begin{cases} \frac{N}{r}(t_{j-1}^N - \theta) & t_j^N \le \theta \le t_{j-1}^N \\ \frac{N}{r}(\theta - t_{j+1}^N) & t_{j+1}^N \le \theta \le t_j^N \\ 0 & \text{otherwise} \end{cases} \qquad j = 1, 2, \ldots N-1$$

$$\phi_0^N(\theta) = \begin{cases} \frac{N}{r}(\theta - t_1^N) & t_1^N \le \theta \le 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_N^N(\theta) = \begin{cases} \frac{N}{r}(t_{N-1}^N - \theta) & -r \le \theta \le t_{N-1}^N \\ 0 & \text{otherwise} \end{cases},$$

where $t_j^N = -j\frac{r}{N}$, $j = 0, 1, 2, \ldots, N$, and define Z_N to be

$$Z_N = \{(\phi(0),\phi) \in Z : \phi = \sum_{j=0}^{N} v_j\phi_j^N \ , \ v_j \in R^n\} \ .$$

Note that $\dim Z_N = n(N+1)$ and that $Z_N \subset D(A)$ as is required by the theory outlined above.

If the $n \times n(N+1)$ matrix function $\Phi^N(\cdot)$ is defined by the relation

$$\Phi^N(\theta) = (\phi_0^N(\theta), \ \phi_1^N(\theta), \ldots, \phi_N^N(\theta)) \otimes I_n$$

for $\theta \in [-r,0]$, where I_n denotes the $n \times n$ identity matrix and $\otimes$ is the Kronecker product, then an arbitrary element in Z_N, $\hat{\psi}^N = (\psi^N(0), \ \psi^N)$ can be represented by

$$\hat{\psi}^N = (\psi^N(0), \ \psi^N) = (\Phi^N(0)\alpha, \ \Phi^N\alpha)$$

for some vector $\alpha \in R^{n(N+1)}$. For an arbitrary element $z = (\eta,\psi) \in Z$ it is shown in [9] that the vector representation α for its projection $P_N z$ with respect to the $\langle\cdot,\cdot\rangle_g$ inner product defined above is given by

(3.7) $\quad \alpha = (K^N)^{-1} h^N(\eta,\psi)$

where the $n(N+1) \times n(N+1)$ nonsingular symmetric matrix K^N is given by

(3.8) $\quad K^N = \Phi^N(0)^T\Phi^N(0) + \int_{-r}^{0} \Phi^N(\theta)^T\Phi^N(\theta)g(\theta)d\theta$

and the mapping $h^N : Z \to R^{n(N+1)}$ is defined by

(3.9) $\quad h^N(\eta,\psi) = \Phi^N(0)^T\eta + \int_{-r}^{0} \Phi^N(\theta)^T\psi(\theta)g(\theta)d\theta$.

This in turn allows for the computation of the matrix representation A^N for the operator A_N,

(3.10) $A^N = (K^N)^{-1}H^N$

where K^N is given in (3.8) and the $n(N+1) \times n(N+1)$ matrix H^N is given by

$$H^N = h^N(L\Phi^N, \dot{\Phi}^N) = \Phi^N(0)^T(L\Phi^N) + \int_{-r}^{0} \Phi^N(\theta)\dot{\Phi}^N(\theta)g(\theta)d\theta \ .$$

For the linear spline case it is not difficult to compute the inner products appearing in the definitions of the matrices K^N and H^N analytically, at least for relatively simple forms of the operator L. The forms for these matrices are given explicitly (in terms of the matrices A_j, $j = 0, 1, 2, \ldots, \nu$, and the matrix function D appearing in the definition of the operator L) in [9] and [10].

In order to compute the matrix representation A^{*N} for the operator A_N^* we note that for arbitrary elements $\hat{\phi}^N$ and $\hat{\psi}^N$ in Z_N with corresponding $n(N+1)$-vector representations α and β respectively, it follows that

$$\text{(3.11)} \quad <\hat{\phi}^N, \hat{\psi}^N>_g = <(\Phi^N(0)\alpha, \Phi^N\alpha) \ , \ (\Phi^N(0)\beta, \Phi^N\beta)>_g$$

$$= \alpha^T\Phi^N(0)^T\Phi^N(0)\beta + \int_{-r}^{0} \alpha^T\Phi^N(\theta)^T\Phi^N(\theta)\beta g(\theta)d\theta$$

$$= \alpha^T K^N \beta$$

where the $n(N+1) \times n(N+1)$ nonsingular matrix K^N is given by (3.8).

Therefore, (3.10) implies that

$$(3.12)\quad \langle A_N\hat{\phi}^N, \hat{\psi}^N\rangle_g = (A^N\alpha)^T K^N\beta = \alpha^T((K^N)^{-1}H^N)^T K^N\beta = \alpha^T(H^N)^T(K^N)^{-1}K^N\beta$$

$$= \alpha^T K^N (K^N)^{-1}(H^N)^T\beta$$

$$= \langle\hat{\phi}^N, \Gamma^N\hat{\psi}^N\rangle_g$$

where Γ^N is the linear transformation on Z_N with matrix representation $(K^N)^{-1}(H^N)^T$. Equation (3.12) implies that $\Gamma^N = A_N^*$ and

$$(3.13)\quad A^{*N} = (K^N)^{-1}(H^N)^T$$

or

$$(3.14)\quad A^{*N} = (K^N)^{-1}(A^N)^T K^N.$$

Since the operators $\mathcal{B}_N : R^m \to Z_N$ and $Q_N : Z_N \to Z_N$ are defined by $\mathcal{B}_N v = P_N(B_0 v, 0)$ and $Q_N(\eta,\psi) = P_N(Q_0\eta, 0)$ respectively, (3.7) and (3.9) imply that their matrix representations, B^N and Q^N are given by

$$(3.15)\quad B^N = (K^N)^{-1}\Phi^N(0)^T B_0$$

and

$$(3.16)\quad Q^N = (K^N)^{-1}\Phi^N(0)^T Q_0\Phi^N(0).$$

Finally, B^{*N}, the matrix representation for the operator $\mathcal{B}_N^*$, can be computed in a manner similar to the one used to compute A^{*N} and is given by

(3.17) $B^{*N} = B_0^T\Phi^N(0)$

or

(3.18) $B^{*N} = (B^N)^T K^N$.

If we let $\tilde{P}^N$ denote the matrix representation for Π_N, the solution to the operator equation (3.1), the matrix form of (3.1) is given by

(3.19) $A^{*N}\tilde{P}^N + \tilde{P}^N A^N - \tilde{P}^N B^N R^{-1} B^{*N}\tilde{P}^N + Q^N = 0.$

Premultiplying by K^N and using (3.14) and (3.18), (3.19) becomes

$$(A^N)^T K^N\tilde{P}^N + K^N\tilde{P}^N A^N - K^N\tilde{P}^N B^N R^{-1}(B^N)^T K^N\tilde{P}^N + K^N Q^N = 0.$$

If the substitutions $P^N = K^N\tilde{P}^N$ and $\tilde{Q}^N = K^N Q^N$ are made in the last equation above, a standard matrix Riccati algebraic equation in $R^{n(N+1)}$ for P^N results and is given by

(3.20) $(A^N)^T P^N + P^N A^N - P^N B^N R^{-1}(B^N)^T P^N + \tilde{Q}^N = 0.$

Equation (3.20) can be solved for the matrix P^N using standard computational techniques and readily available software packages (see [2] and [20]).

Once P^N has been determined, a simple calculation reveals that the N^{th} approximating optimal control for our problem given by (3.4) takes the form

(3.21) $\bar{u}_N(t) = -[\, R^{-1}B_0^T\Phi^N(0)(K^N)^{-1}P^N(K^N)^{-1}\Phi^N(0)^T x(t)$

$$+ \int_{-r}^{0} R^{-1}B_0^T\Phi^N(0)(K^N)^{-1}P^N(K^N)^{-1}\Phi^N(\theta)^T x(t+\theta)g(\theta)d\theta].$$

Comparing (3.21) to (3.4), it is immediately clear that the approximating feedback gains are given by

$$(3.22)\quad R^{-1}B_0^T\Pi_N^{00} = R^{-1}B_0^T\Phi^N(0)(K^N)^{-1}P^N(K^N)^{-1}\Phi^N(0)^T$$

and

$$(3.23)\quad R^{-1}B_0^T\Pi_N^{10}(\cdot)^T = R^{-1}B_0^T\Phi^N(0)(K^N)^{-1}P^N(K^N)^{-1}\Phi^N(\cdot)^T g(\cdot)\ .$$

Using (3.6) and (3.11) we obtain an approximation to the optimal value of the cost functional. For a given set of initial conditions $x(0) = \eta$, $x_0 = \psi$ we have

$$(3.24)\quad J(\bar{u};\eta,\psi) \sim [(K^N)^{-1}h^N(\eta,\psi)]^T P^N(K^N)^{-1}h^N(\eta,\psi)$$

where h^N is given by (3.9).

The approximation scheme which was developed above is semi-discrete in nature in that the approach taken is based primarily upon the approximation of the infinite dimensional state equation (2.7) in the space Z by a sequence of finite dimensional ordinary differential equations in Z_N of the form (3.2). However it is also possible to develop a parallel theory which is based upon a full discretization of the optimization problem in the spirit of the results presented in [23]. The cost functional (2.6) is discretized and the state equation (in its integrated form (2.9)) is approximated by a finite dimensional difference equation in Z_N resulting in a finite dimensional discrete steady state linear regulator problem which can be solved in feedback form using conventional methods. We sketch briefly the particulars of such an approach.

Let the N^{th} approximating optimization problem be given by:

Find a sequence $\{\bar{u}_t^N\}_{t=0}^{\infty}$ of m-vectors in ℓ_2 such that $\bar{u}_t^N$ minimizes

$$J^N(\{u_t^N\};\, z_0) = \sum_{t=0}^{\infty} \langle \hat{Q}_N z_t^N,\, z_t^N\rangle + \langle R_N u_t^N,\, u_t^N\rangle$$

subject to

$$(3.25)\quad z_{t+1}^N = P_{ij}(\tfrac{r}{N}A_N)\, z_t^N + \tfrac{r}{N}P_{k\ell}\,(\tfrac{r}{2N}A_N)\, \mathcal{B}_N u_t^N \qquad t = 1, 2, \ldots$$

$$(3.26)\qquad z_0^N = \mathcal{P}_N z_0$$

where $\hat{Q}_N = \frac{r}{N} Q_N$, $R_N = \frac{r}{N}R$ and A_N, $\mathcal{P}_N$, $\mathcal{B}_N$, Q_N, R and z_0 are as they have been defined above. The rational functions $P_{ij}(z)$ and $P_{k\ell}(z)$ are selected from among the entries in the diagonal or first two subdiagonals of the Padé table of rational function approximations to the exponential.

The basis for the construction of the approximation problems is the fact that the variation of parameters form of the solution to (3.25), (3.26) given by

$$z_t^N = P_{ij}(\tfrac{r}{N}A_N)^t \mathcal{P}_N z_0 + \frac{r}{N} \sum_{j=1}^{t} P_{ij}(\tfrac{r}{N}A_N)^{t-s} P_{k\ell}(\tfrac{r}{2N}A_N)\, \mathcal{B}_N u_s^N$$

is an approximation to (2.9) in the sense that

$$|z_t^N - z(\tfrac{tr}{N})| \to 0$$

an $N \to \infty$ uniformly in t for $t \in \{0, 1, 2 \ldots [\frac{t_f N}{r}]\}$ for any $t_f < \infty$, where the symbol $[\alpha]$ denotes the greatest integer less than or equal to α (see [23]).

The feedback form of the solution (if it exists) to the corresponding approximating problem and the optimal value of the cost functional are given by

$$\bar{u}_t^N = - F_N z_t^N \qquad t = 0, 1, 2, \ldots ,$$

and

(3.27) $J^N(\{\bar{u}_t^N\}; z_0) = \langle \Pi_N P_N z_0, P_N z_0 \rangle$,

respectively, where the linear operators $F_N : Z_N \to R^m$ and $\Pi_N : Z_N \to Z_N$ are determined by solving the system of operator equations (see [20])

(3.28) $\Pi_N = X_N^* \Pi_N X_N + F_N^* R_N F_N + \hat{Q}_N$

(3.29) $X_N = \tilde{A}_N - \tilde{B}_N F_N$

(3.30) $F_N = (R_N + \tilde{B}_N^* \Pi_N \tilde{B}_N)^{-1} \tilde{B}_N^* \Pi_N \tilde{A}_N$

in the unknowns Π_N, X_N and F_N where $\tilde{A}_N = P_{ij}(\frac{r}{N} A_N)$ and $\tilde{B}_N = \frac{r}{N} P_{k\ell}(\frac{r}{N} A_N) B_N$.

To actually compute the optimal control law, the system (3.28), (3.29), (3.30) must first be transformed into an equivalent matrix formulation. Adopting the convention that the symbol T^N will denote the matrix representation for the operator T_N with respect to the linear spline basis defined above, it is not difficult to show that the system (3.28), (3.29), (3.30) is equivalent to the system of matrix equations given by

(3.31) $P^N = (X^N)^T P^N X^N + (F^N)^T R^N F^N + \frac{r}{N} K^N Q^N$

$$(3.32) \quad X^N = \tilde{A}^N - \tilde{B}^N F^N$$

$$(3.33) \quad F^N = (R^N + (\tilde{B}^N)^T P^N \tilde{B}^N)^{-1} (\tilde{B}^N)^T P^N \tilde{A}^N$$

where $P^N = K^N \Pi^N$, $\tilde{A}^N = P_{ij}(\frac{r}{N} A^N)$, $\tilde{B}^N = \frac{r}{N} P_{k\ell}(\frac{r}{N} A^N) B^N$, $R^N = \frac{r}{N} R$ and K^N, A^N, B^N and Q^N are given by (3.8), (3.10), (3.15) and (3.16) respectively. Standard software packages can be used to solve the system (3.31), (3.32), (3.33), see for instance [2]. Once the matrices F^N and P^N have been determined, the N^{th} approximating solution to our problem is given by

$$(3.34) \quad \bar{u}_N(t) = - F_N P_N z(t)$$

$$= - [F^N (K^N)^{-1} \Phi^N(0)^T x(t) + \int_{-r}^{0} F^N (K^N)^{-1} \Phi^N(\theta)^T x(t+\theta) g(\theta) d\theta]$$

or using the fact that $\Pi_N P_N$ approximates Π, by

$$(3.35) \quad \bar{u}_N(t) = - R^{-1} B^* \Pi_N P_N z(t)$$

$$= - [R^{-1} B_0^T (K^N)^{-1} P^N (K^N)^{-1} \Phi^N(0)^T x(t)$$

$$+ \int_{-r}^{0} R^{-1} B_0^T (K^N)^{-1} P^N (K^N)^{-1} \Phi^N(\theta)^T x(t+\theta) g(\theta) d\theta].$$

The feedback gains K_0 and K_1 in (2.4) are approximated by

$$(3.36) \quad F^N (K^N)^{-1} \Phi^N(0)^T$$

and

$$(3.37) \quad F^N (K^N)^{-1} \Phi^N(\cdot)^T g(\cdot) ,$$

respectively, if (3.34) is used and by

(3.38) $R^{-1}B_0^T(K^N)^{-1}P^N(K^N)^{-1}\Phi^N(0)^T$

and

(3.39) $R^{-1}B_0^T(K^N)^{-1}P^N(K^N)^{-1}\Phi^N(\cdot)^T g(\cdot)$

if (3.35) is used.

Finally for a given set of initial conditions $x(0) = \eta$, $x_0 = \psi$, the optimal value of the cost functional can be approximated using (3.27). From (3.11) we find

(3.40) $J(\bar{u};\, \eta,\psi) \sim [(K^N)^{-1}h^N(\eta,\psi)]^T P^N(K^N)^{-1}h^N(\eta,\psi)$

where h^N is given by (3.9).

4. NUMERICAL RESULTS

In this section we present, discuss and analyze numerical results obtained by using the linear spline based approximation schemes described above to compute feedback controls for several hereditary regulator problems of the form given in section 2. For the purpose of comparison we have also computed approximate solutions using the finite difference based AVE scheme discussed in [14]. We recall that for the semi-discrete spline scheme, the approximating feedback gains $R^{-1}B_0^T\Pi_N^{00}$ and $R^{-1}B_0^T\Pi_N^{10}(\cdot)^T$ are given by (3.22) and (3.23) respectively while for the AVE scheme they may be computed using the time invariant forms of (7.27) and (7.28) in [14]. For the fully discrete spline scheme, the approximating gains are given either by (3.36) and (3.37) or by (3.38) and (3.39). Analogous formulae can be derived for a fully discrete scheme based upon the AVE approximation.

All computations were performed on a Control Data Corporation Cyber 170 model 730 at the NASA Langley Research Center (LaRC) using software written in Fortran. For the semi-discrete schemes, the approximating matrix Riccati algebraic equations (3.20) were solved using both an iterative Newton technique as it is described in [16] and the Potter method (see [22],[20]) which involves the eigenvalue-eigenvector decomposition of the $2n(N+1) \times 2n(N+1)$ matrix

$$\hat{A}^N = \begin{bmatrix} A^N & -B^N R^{-1} B^{N^T} \\ -\tilde{Q}^N & -A^{N^T} \end{bmatrix}$$

where the matrices A^N, B^N and $\tilde{Q}^N$ are as they were defined in section 3. The implementations of the two methods we used are contained in ORACLS [2], a software package developed at LaRC for the design of multivariable control systems.

Although the Newton algorithm performed well on the equations arising from both the spline and AVE schemes, the Potter method was the more efficient of the two, particularly for large values of N. The ORACLS implementation of the Potter method however, requires that the matrix $\hat{A}^N$ be diagonalizable. This additional requirement posed no difficulties for the spline schemes in any of the examples we considered. On the other hand, for the AVE scheme, certain classes of problems (including those involving state equations of dimension greater than one of the form considered in Examples 4.2, 4.3 and 4.4 below) lead to $\hat{A}^N$ which are non-diagonalizable (see [14] Theorems 7.11 and 7.12). In this instance, if one wishes to use the Potter method to solve (3.20), the generalized eigenvectors of $\hat{A}^N$ must be computed.

For the fully-discrete schemes, the system (3.31), (3.32), (3.33) was solved using an iterative Newton algorithm from the ORACLS package.

We have included results for five examples. Example 4.1 involves a 1 dimensional state equation while Examples 4.2 and 4.3 involve systems of dimension 2. In Example 4.4 we consider the wind tunnel system described in section 1. Examples 4.1 - 4.4 were all solved using semi-discrete approximations, while in Example 4.5 the fully-discrete method was used to solve the scalar problem considered in Example 4.1.

Example 4.1

We consider the minimization of

$$(4.1)\qquad J(u;x(0),x_0) = \int_0^\infty [x^2(t) + u^2(t)]dt$$

subject to the scalar differential equation given by

$$(4.2)\qquad \dot{x}(t) = x(t) + x(t-1) + u(t).$$

In this example, the Π_N^{00} are scalars and have been tabulated for N = 4, 8, 16, and 32 in Table 4.1. The $\Pi_N^{10}(\cdot)$ are scalar valued functions and have been plotted for the same values of N in Figures 4.1, 4.2, 4.3 and 4.4.

N	AVE	SPLINE
4	2.8866	2.7940
8	2.8476	2.8054
16	2.8278	2.8084
32	2.8182	2.8091

TABLE 4.1

Although we do not have a true value for Π^{00}, it is immediately clear from Table 4.1 that the values computed using the spline based scheme appear to have converged, while those computed by the AVE scheme are converging much more slowly. The oscillatory behavior exhibited by the spline approximations to Π^{10} is a consequence of the fact that while in general it is not the case that $\Pi^{00} = \Pi^{10}(0)$, the requirement $R(\Pi_N) \subset Z_N \subset D(A)$ imposes the conditions $\Pi_N^{00} = \Pi_N^{10}(0)$ for each N. However, because in the closed loop form of the state equation (see (3.5)) $\Pi_N^{10}(\cdot)$ appears in the form of the kernel of an integral operator, the effect of the oscillations is minimized.

We selected the initial data

$$(4.3) \qquad x(0) = 0 \qquad x_0(\theta) = \sin \pi\theta \, , \quad -1 \le \theta \le 0 \, ,$$

and computed the trajectories which result when the approximating optimal feedback controls (computed using either the AVE or the spline approximation schemes) are applied to the system (4.2), (4.3). Approximate values for the cost functional (4.1) were computed two ways: directly using the approximating

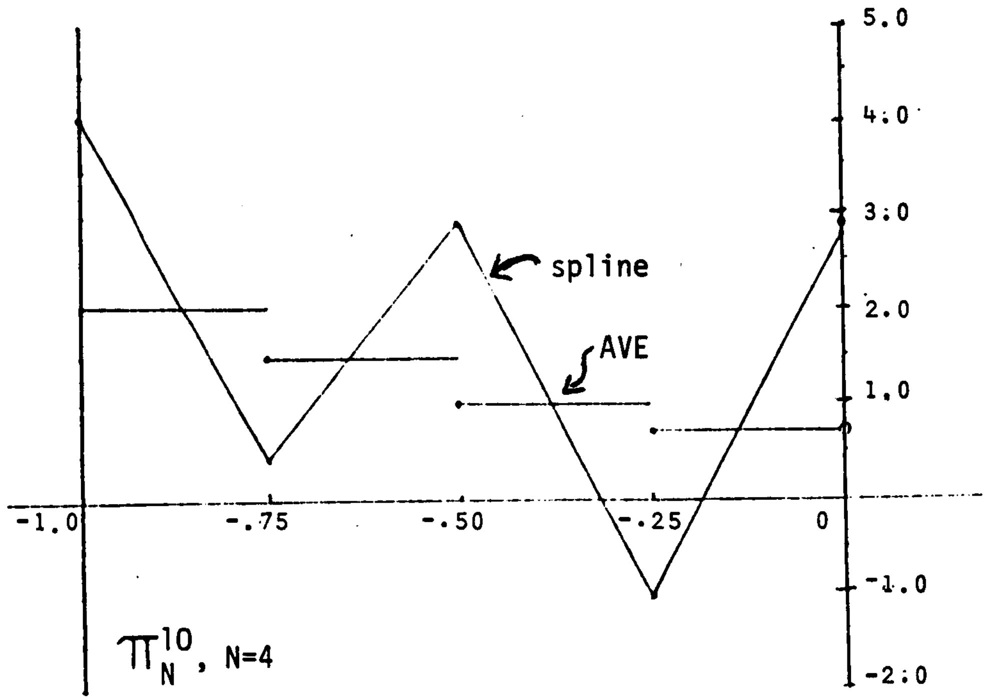

FIGURE 4.1

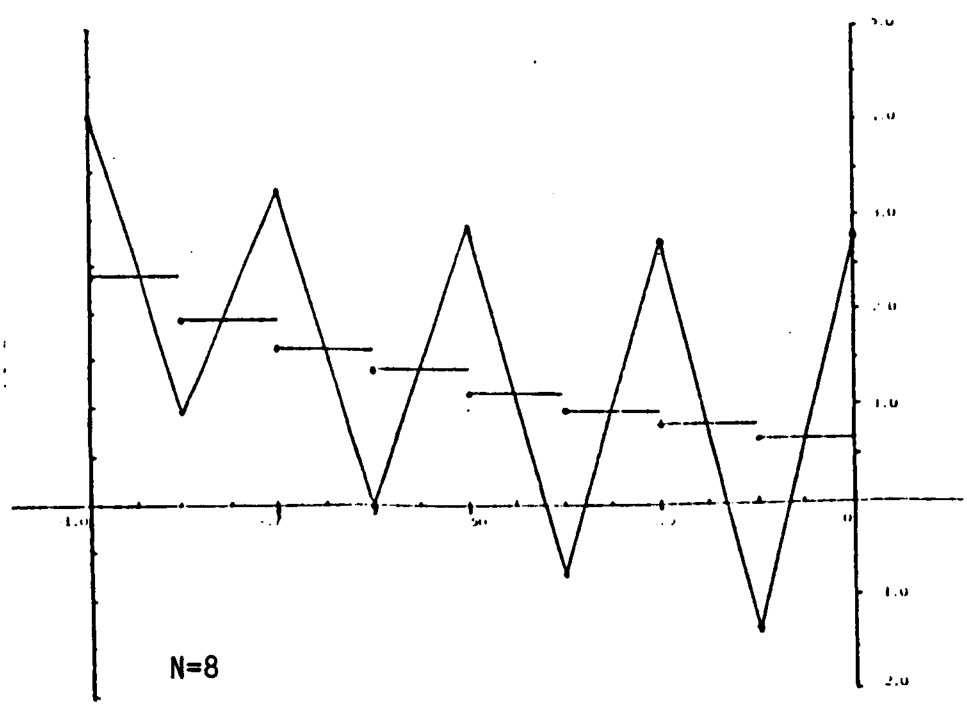

FIGURE 4.2

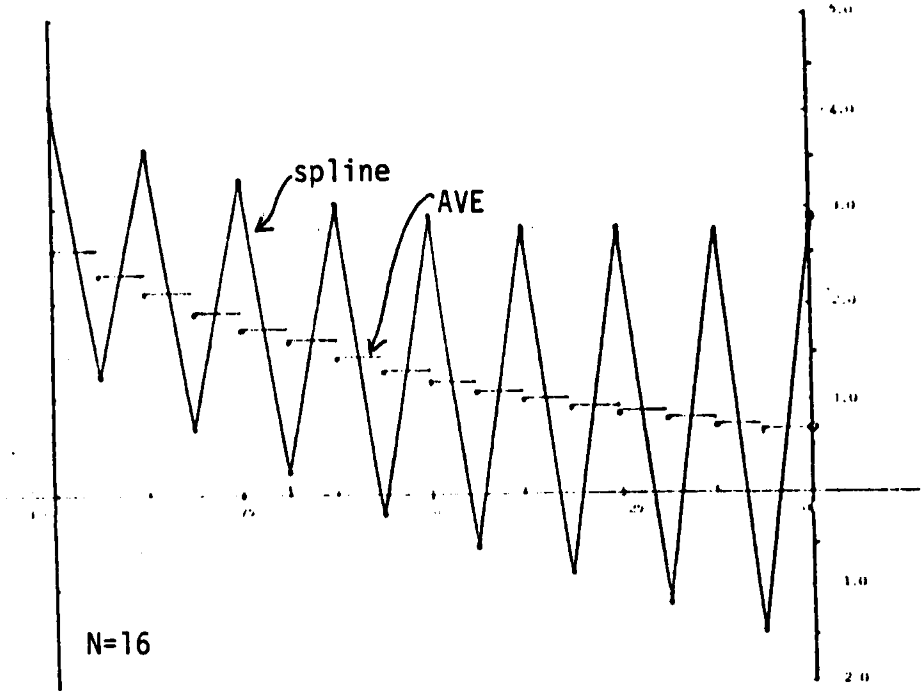

FIGURE 4.3

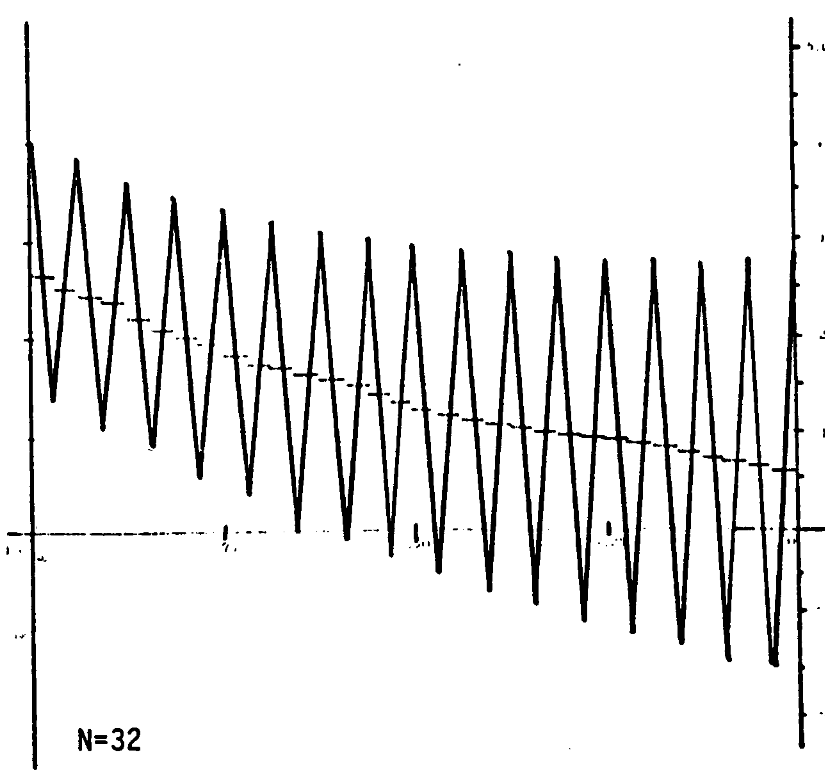

FIGURE 4.4

optimal controls (3.4) with the corresponding approximating optimal trajectories (3.5) in (4.1) and also via (3.24). The integration of the closed loop state equation (3.5) was carried out by first discretizing the integral term and then applying a modified version of a Runge-Kutta method for the numerical solution of ordinary differential equation initial value problems. We note that the numerical integration method employed to compute these trajectories was completely independent of either of the approximation schemes used to compute the approximating feedback operators, and thus should not have biased our results. For N = 4, 8, 16, and 32 the approximating optimal trajectories are plotted in Figures 4.5, 4.7, 4.9 and 4.11 while the open loop form of the approximating optimal controls are plotted in Figures 4.6, 4.8, 4.10 and 4.12. The approximating values for the cost functional are tabulated in Table 4.2 where columns 1 and 3 contain the values computed directly and columns 2 and 4 the values computed using (3.24).

N	AVE		SPLINE	
	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$
4	.3309	.2809	.3272	.2484
8	.3281	.3000	.3271	.3027
16	.3275	.3121	.3272	.3163
32	.3274	.3191	.3273	.3196

TABLE 4.2

We have also computed trajectories and controls for the system (4.2) with the initial data

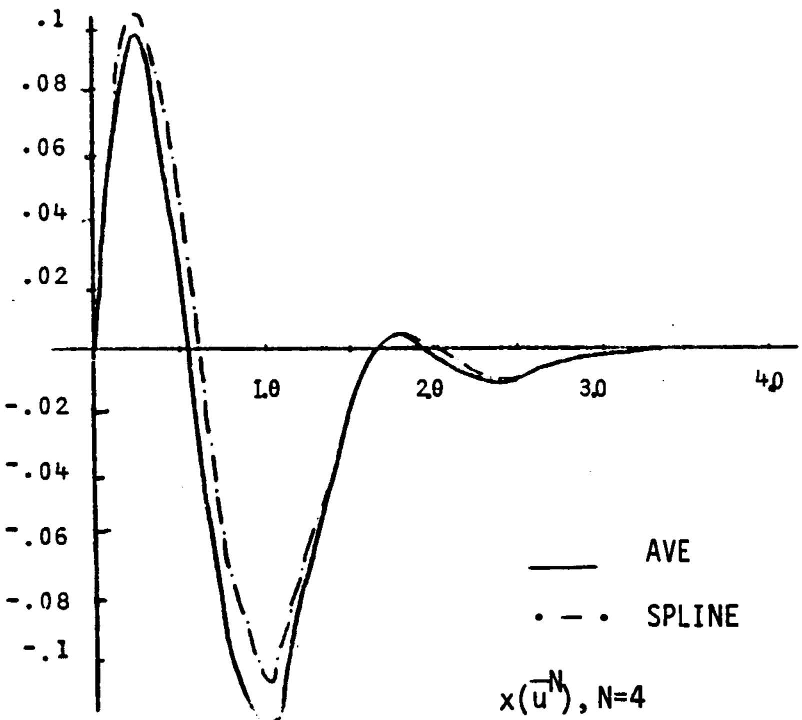

FIGURE 4.5

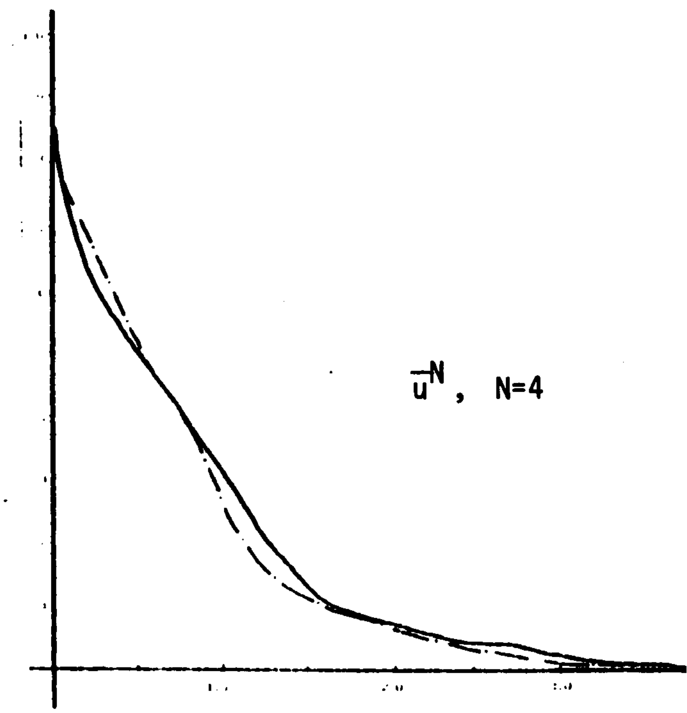

FIGURE 4.6

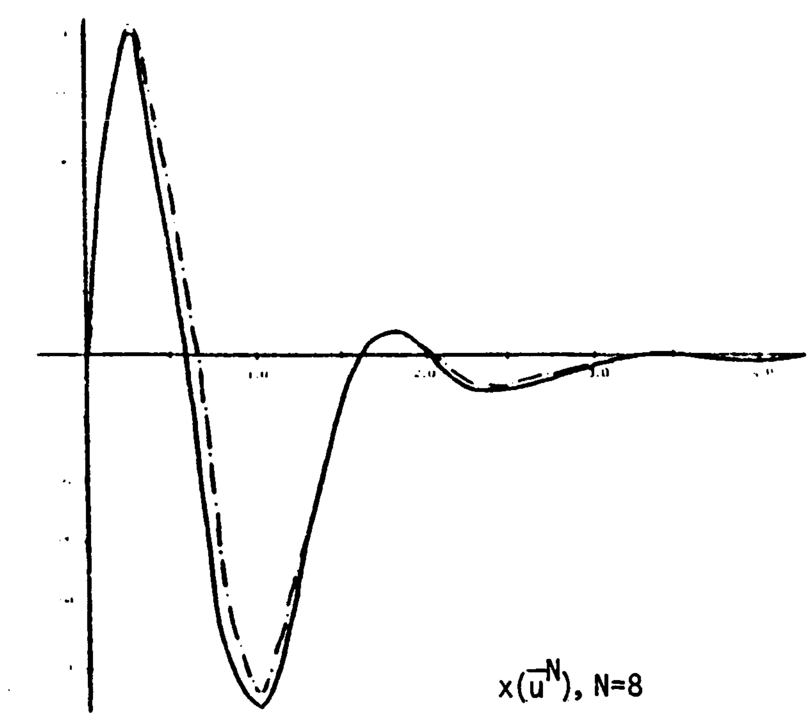

FIGURE 4.7

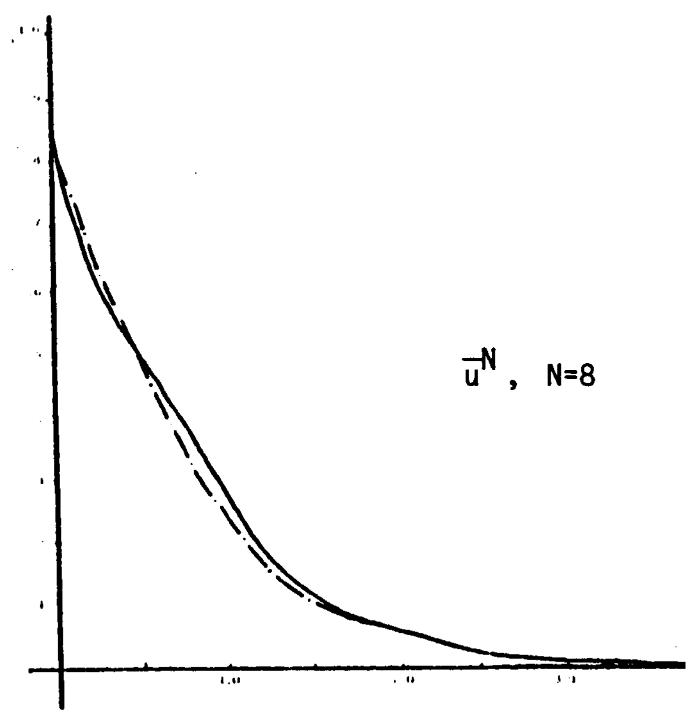

FIGURE 4.8

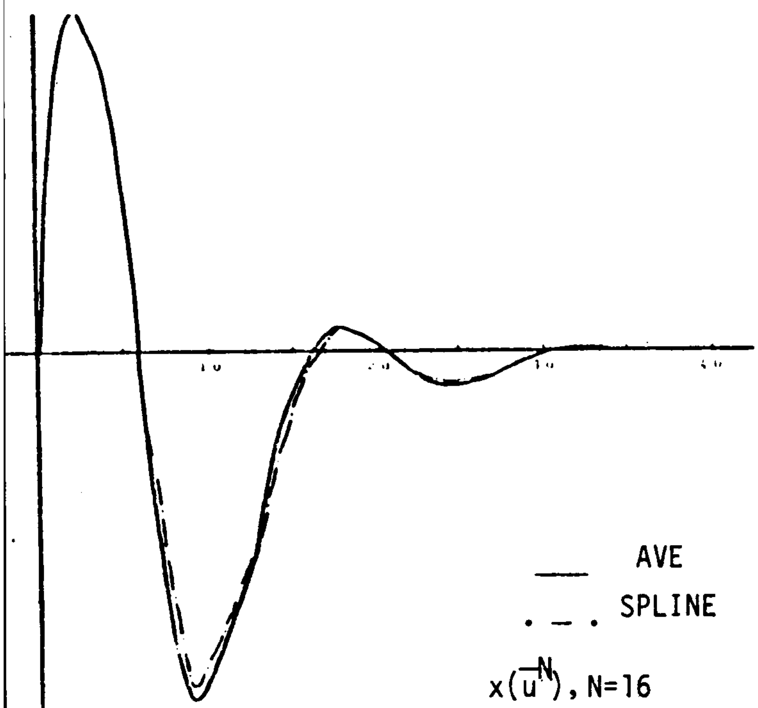

FIGURE 4.9

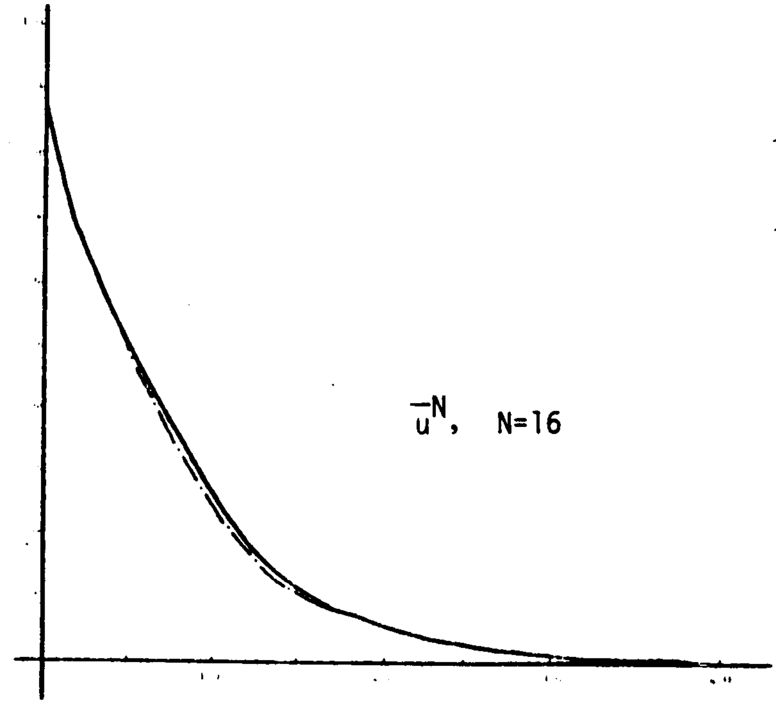

FIGURE 4.10

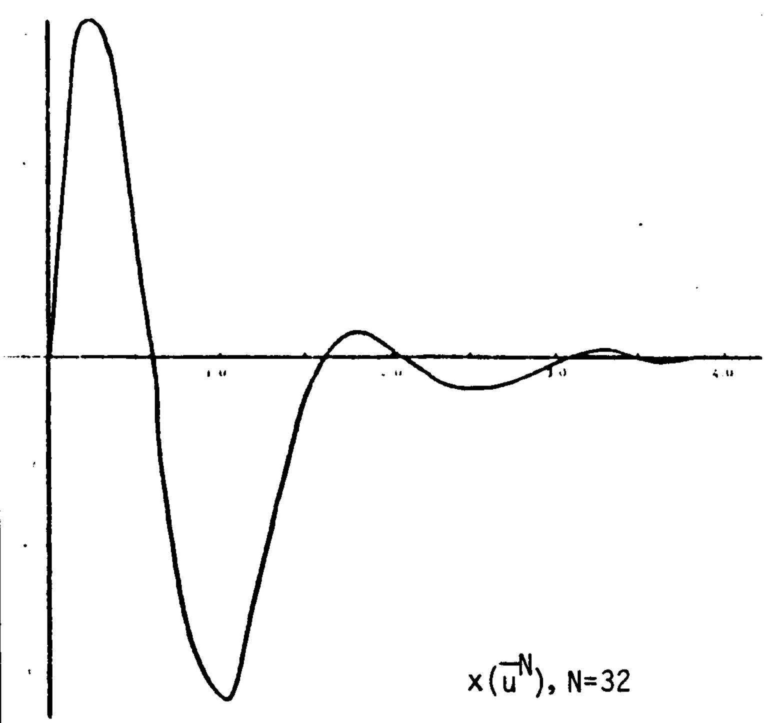

FIGURE 4.11

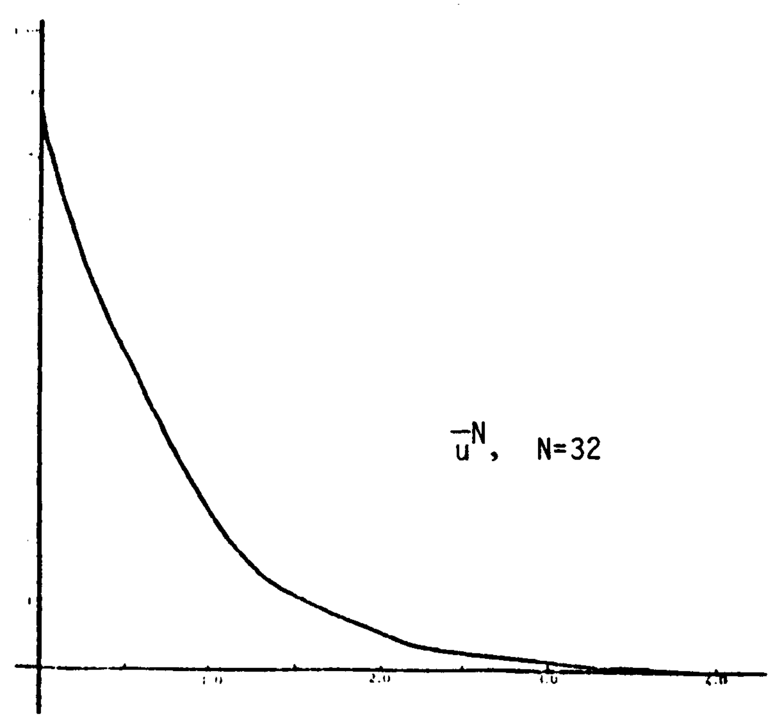

FIGURE 4.12

$$x(0) = 0 \qquad x_0(\theta) = \begin{cases} 2(\theta+1) & -1 \le \theta \le -\frac{1}{2} \\ -2\theta & -\frac{1}{2} \le \theta \le 0 . \end{cases}$$

The resulting values of the cost functional (4.1) are given in Table 4.3.

N	AVE		SPLINE	
	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$
4	.2036	.1724	.2015	.1972
8	.2017	.1840	.2012	.1972
16	.2013	.1914	.2012	.1974
32	.2012	.1959	.2012	.1975

TABLE 4.3

Based upon the numerical results for the examples presented above and several others which we have considered, the following observations concerning the relative performance of the AVE and Spline based schemes can be made.

(I) The spline scheme converges faster and is more accurate at low orders. The AVE approximations generate a scheme that appears to converge like $1/N$ while that for the splines is like $1/N^2$. This is not unexpected, given our experience with the AVE and spline approximations in other contexts (e.g., see [5], [9], [7]). Furthermore, the trajectories, controls, and cost functional values obtained using the spline approximations with N = 4 are competitive with the results produced by the AVE scheme with N = 16. The computational effort and expense involved in solving a high order Riccati equation makes this an important consideration. In the scalar examples we tested the amount of CPU time required to solve (using the Newton algorithm) the 5 dimensional Riccati equations (corresponding to $n = 1$ and $N = 4$) was on the order of 10 seconds while the 17 dimensional equations ($n = 1$, $N = 16$) required approximately 70 seconds.

(II) If the value of the cost functional J computed using the approximating optimal control and the resulting trajectory is used as a measure of the relative performance of the two approximation methods, then we found that the spline based technique is preferable. The spline approximations consistantly produced a smaller value for $J(\bar{u}_N;x(0),x_0)$ than did the AVE scheme.

Similar conclusions can be drawn in the case of higher dimensional equations. Numerical results for two second order equations are presented in Examples 4.2 and 4.3 below. Since the qualitative behavior of each component of the matrix valued functions $\Pi_N^{10}(\cdot)$ was the same as already depicted here in the 1 dimensional cases, for the 2 dimensional examples we present only the computed values for the 2×2 matrices Π_N^{00}.

Example 4.2

We consider the problem of minimizing

$$(4.4) \quad J(u;y(0),y_0,\dot{y}(0),\dot{y}_0) = \int_0^\infty [y(t)^2 + \dot{y}(t)^2 + u(t)^2]dt$$

subject to the harmonic oscillator with delayed restoring force given by

$$(4.5) \quad \ddot{y}(t) + y(t-1) = u(t).$$

If we define x(t) by

$$x(t) = \begin{bmatrix} y(t) \\ \\ \dot{y}(t) \end{bmatrix},$$

then (4.4) and (4.5) are equivalent to

$$J(u;x(0),x_0) = \int_0^\infty [x(t)^T x(t) + u(t)^2]dt$$

and

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t-1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

respectively.

The values for Π_N^{00} with N = 4, 8, and 16 obtained using the AVE and spline schemes are given in Table 4.4 below.

N	AVE	SPLINE
4	$\begin{bmatrix} 2.8179 & 1.2816 \\ 1.2816 & 1.8876 \end{bmatrix}$	$\begin{bmatrix} 3.0675 & 1.4222 \\ 1.4222 & 1.9590 \end{bmatrix}$
8	$\begin{bmatrix} 2.9276 & 1.3454 \\ 1.3454 & 1.9211 \end{bmatrix}$	$\begin{bmatrix} 3.0658 & 1.4210 \\ 1.4210 & 1.9596 \end{bmatrix}$
16	$\begin{bmatrix} 2.9920 & 1.3814 \\ 1.3814 & 1.9398 \end{bmatrix}$	$\begin{bmatrix} 3.0655 & 1.4207 \\ 1.4207 & 1.9599 \end{bmatrix}$

TABLE 4.4

Table 4.5 contains (in columns 1 and 3) the values of the cost functional (4.4) which were obtained when the approximating feedback controllers were applied to the system (4.5) together with the initial conditions

$$y(0) = 1 \qquad y(\theta) = 1 \qquad -1 \le \theta \le 0$$

$$\dot{y}(0) = 0 \qquad \dot{y}(\theta) = 0 \qquad -1 \le \theta \le 0 .$$

N	AVE		SPLINE	
	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$
4	3.2861	3.1700	3.2750	3.3857
8	3.2771	3.2165	3.2748	3.3693
16	3.2751	3.2407	3.2747	3.3659

TABLE 4.5

Example 4.3

In this example we again consider the minimization of (4.4), however this time subject to the harmonic oscillator with delayed restoring force and delayed damping given by

$$(4.6) \qquad \ddot{y}(t) + \dot{y}(t-1) + y(t-1) = u(t)$$

or, equivalently

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} x(t-1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

where $x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$.

The values of Π_N^{00} for this example are given in Table 4.6 below.

N	AVE	SPLINE
4	$\begin{bmatrix} 1.9849 & 1.1248 \\ 1.1248 & 1.6538 \end{bmatrix}$	$\begin{bmatrix} 2.1419 & 1.2952 \\ 1.2952 & 1.853528 \end{bmatrix}$
8	$\begin{bmatrix} 2.0511 & 1.1991 \\ 1.1991 & 1.7432 \end{bmatrix}$	$\begin{bmatrix} 2.1394 & 1.296 \\ 1.296 & 1.8568 \end{bmatrix}$
16	$\begin{bmatrix} 2.0914 & 1.2440 \\ 1.2440 & 1.7965 \end{bmatrix}$	$\begin{bmatrix} 2.1389 & 1.2963 \\ 1.2963 & 1.8576 \end{bmatrix}$

TABLE 4.6

We note that using the AVE scheme, Gibson [14] computed Π_{22}^{00} to be

$$\begin{bmatrix} 2.1034 & 1.2574 \\ 1.2574 & 1.8123 \end{bmatrix}$$

with the convergence being monotonic from below in each component of the matrix.

The values of the cost functional (4.4) evaluated using the trajectories obtained by integrating (4.5) with u given by (3.4) and initial conditions

$$y(0) = 0 \qquad y(\theta) = \begin{cases} 2(\theta+1) & -1 \le \theta \le -\frac{1}{2} \\ -2\theta & -\frac{1}{2} \le \theta \le 0 \end{cases}$$

$$\dot{y}(0) = -2 \qquad \dot{y}(\theta) = \begin{cases} 2 & -1 \le \theta \le -\frac{1}{2} \\ -2 & -\frac{1}{2} \le \theta \le 0 \end{cases}$$

are given in Table 4.7.

N	AVE		SPLINE	
	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$
4	14.4103	11.2650	14.2383	13.8926
8	14.2493	12.3886	14.2201	13.8045
16	14.2165	13.0784	14.2160	13.8308

TABLE 4.7

Example 4.4

In this example we investigate the Mach no. control loop problem described in the introduction. Recall that when the guide vane angle actuator is the control, the state of the system is governed by an equation in R^3 of the form

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{\tau} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & -2\zeta\omega \end{bmatrix} x(t) + \begin{bmatrix} 0 & \frac{k}{\tau} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t-.33) + \begin{bmatrix} 0 \\ 0 \\ \omega^2 \end{bmatrix} u(t).$$

Here $x = (\delta M, \delta\theta, \delta\dot{\theta})^T$ is made up of the variation in Mach no. δM, the variation in guide vane angle $\delta\theta$, and the variation in guide vane angle velocity $\delta\dot{\theta}$, $u = \delta\theta_A$ is the guide vane angle actuator input and the parameters τ, ω, ζ, and k take on the values 1.964 sec., 6.0 rad/sec., .8, and -.0117 deg^{-1} respectively [3].

Parametric studies [3] on the elements of the state weighting matrix in the cost functional revealed that if J was chosen as

$$J(u;x(0),x_0) = \int_0^\infty [x(t)Q_0x(t) + u^2(t)]dt$$

where $Q_0 = \mathrm{diag}(10^4,0,0)$, then the resulting control gains, upon simulation, produced responses which typically did not exceed the physical limitations of the system.

The values for the matrices Π_N^{∞} computed using the AVE and spline schemes for N = 2, 4, and 8 are given in Table 4.8. For the AVE scheme with N = 8, the Newton iteration did not converge to a solution of the matrix Riccati equation. Once again the values computed using the spline approach appear to have converged.

N	AVE	SPLINE
2	$\begin{bmatrix} 26228.0114 & -29.9732 & -2.8990 \\ -29.9732 & .0515 & .0052 \\ -2.8990 & .0052 & .0005 \end{bmatrix}$	$\begin{bmatrix} 26314.4858 & -29.7940 & -2.8708 \\ -29.7940 & .0560 & .0056 \\ -2.8708 & .0056 & .0006 \end{bmatrix}$
4	$\begin{bmatrix} 26259.9665 & -29.8569 & -2.8858 \\ -29.8569 & .0532 & .0054 \\ -2.8858 & .0054 & .0006 \end{bmatrix}$	$\begin{bmatrix} 26296.0999 & -29.7558 & -2.8712 \\ -29.7558 & .0561 & .0056 \\ -2.8712 & .0056 & .0006 \end{bmatrix}$
8	———	$\begin{bmatrix} 26294.1091 & -29.7459 & -2.8716 \\ -29.7459 & .0561 & .0056 \\ -2.8716 & .0056 & .0006 \end{bmatrix}$

TABLE 4.8

Using the approximating feedback control laws we computed trajectories for the problem of driving δM from -.1 to 0.0 (corresponding to M varying from .8 to .9) and $\delta\theta$ from 8.55 to 0 (corresponding to the guide vane angle varying from 10.48^{o} to a steady state of 1.93^{o}). The initial variation in the guide vane angle velocity, $\dot{\delta\theta}(0)$ was set to 0. The resulting values for the cost functional are given in Table 4.9. The Mach no. and guide vane angle trajectories produced by the approximating control gains computed using the spline scheme with N = 8 are plotted in Figures 4.13 and 4.14, respectively. Our results compare favorably with those obtained by Armstrong and Tripp [3] using an approximating feedback control law produced by a finite difference technique and with those obtained by Daniel [11] using a spline based approximation scheme to solve a similar problem in open loop form.

N	AVE		SPLINE	
	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$
2	414.371	408.0646	414.350	435.8053
4	414.3612	410.6452	414.3496	435.9128
8	—	—	414.3495	436.0064

TABLE 4.9

<u>Example 4.5</u>

In this example we apply the fully discrete method discussed at the end of section 3 to the scalar problem considered in Example 4.1. The feedback gains were computed by solving the system (3.31), (3.32), (3.33) where the matrices $\tilde{A}^N$ and $\tilde{B}^N$ were constructed using the (9, 9) Padé approximant together

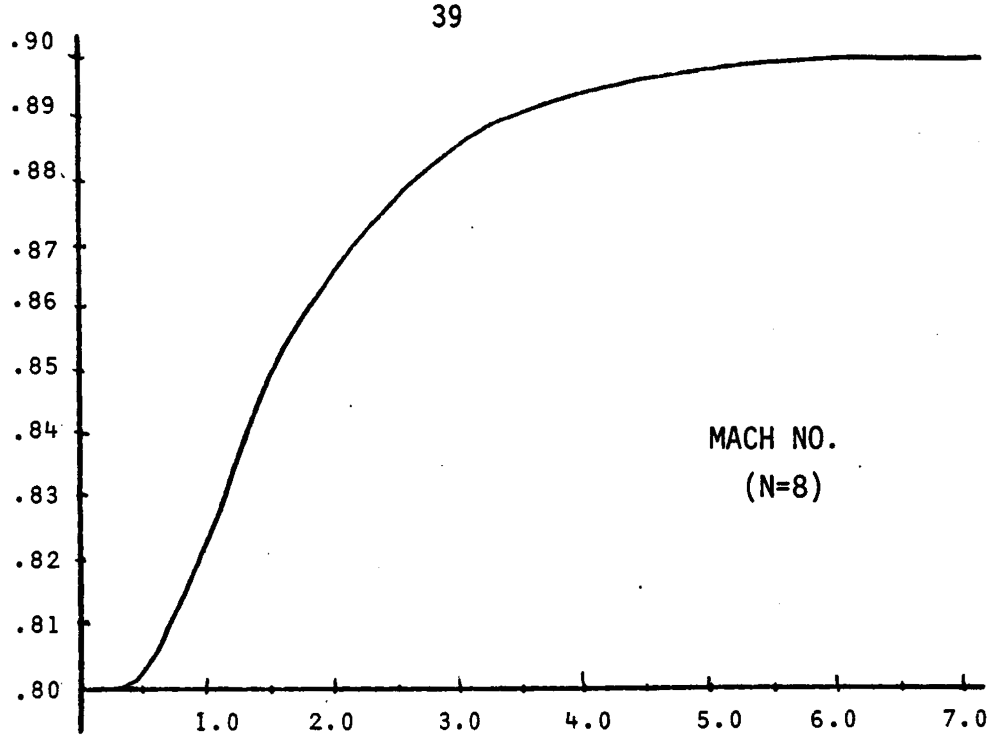

FIGURE 4.13

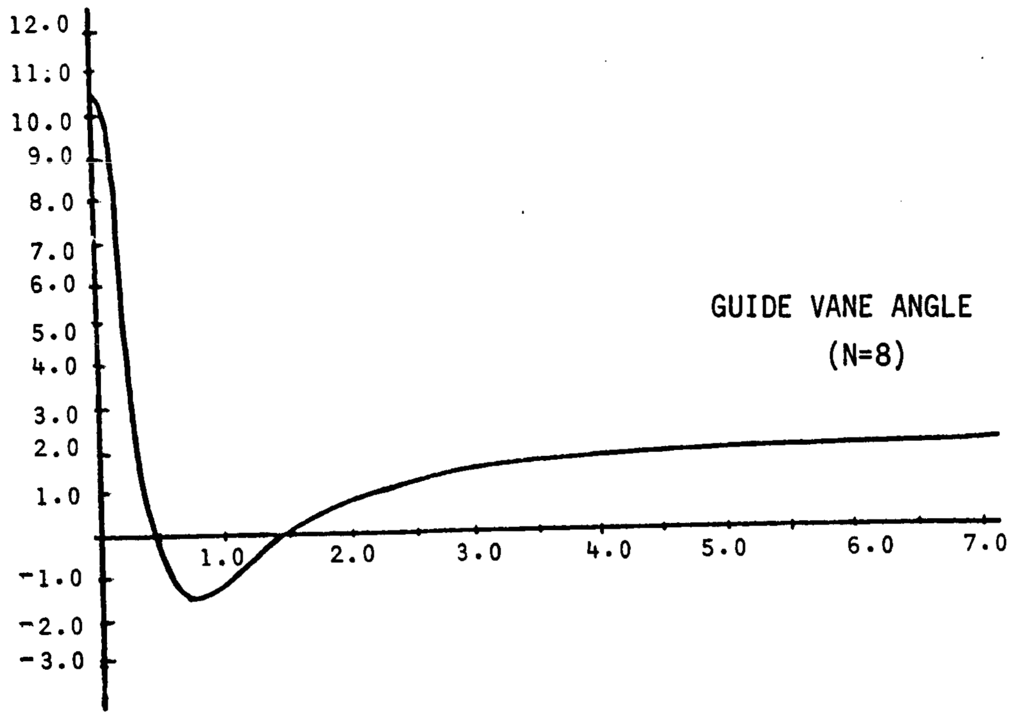

FIGURE 4.14

with the matrix representation for the operator A_N with respect to either the AVE or linear spline basis. The matrices $P_{99}(\frac{r}{N}A^N)$ were computed using the routine EXPADE in the ORACLS library.

Approximating feedback gains were computed using both the operator $F_N P_N$, as in (3.34), and the operator $R^{-1}B^{*}\Pi_N P_N$ as in (3.35). The resulting approximations to K_0 in (2.4) are given in Table 4.10 and Table 4.11, respectively.

N	AVE	SPLINE
4	2.4086	2.3437
8	2.5898	2.5550
16	2.6940	2.6764
32	2.7500	2.7414

TABLE 4.10

N	AVE	SPLINE
4	3.0763	2.9556
8	2.9257	2.8768
16	2.8629	2.8418
32	2.8348	2.8253

TABLE 4.11

The approximations to K_1 behaved qualitatively in a manner similar to their behavior in the semi-discrete case; oscillatory for the linear spline scheme, and smooth for the AVE scheme (see Figures 4.1 - 4.4). For both K_0 and K_1 the convergence appears to be slower for the fully discrete schemes than for the semi-discrete schemes.

In order to compare the relative performance of the fully discrete and semi-discrete methods, the two control laws, (3.34) and (3.35), and the spline and AVE schemes, we have computed trajectories corresponding to the initial conditions (4.3) and tabulated the resulting values of the cost functional computed both directly and using (3.40). Our results corresponding to control law (3.34) are given in Table 4.12 while those obtained using control law (3.35) are given in Table 4.13.

N	AVE		SPLINE	
	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$
4	.3458	.2859	.3344	.2520
8	.3309	.3013	.3283	.3039
16	.3278	.3124	.3273	.3166
32	.3273	.3192	.3272	.3197

TABLE 4.12

N				
	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$	$J(x(\bar{u}_N))$	$\langle \Pi_N P_N z_0, P_N z_0 \rangle$
4	.3314	.2859	.3276	.2520
8	.3283	.3013	.3273	.3039
16	.3275	.3124	.3273	.3166
32	.3274	.3192	.3273	.3197

TABLE 4.13

Our numerical results for the fully discrete schemes appear to point to the following conclusions:

(I) For low order approximation (i.e., N small) the results produced by the semi-discrete schemes are better than those produced by the fully discrete methods. As N increases, however, the two techniques yield comparable results.

(II) For N large, using an iterative Newton algorithm, the ORACLS package was able to solve the system (3.31),(3.32),(3.33) arising in the fully discrete methods in roughly half the time it required to solve the matrix Riccati algebraic equation (3.20) resulting from the semi-discrete approximation schemes.

(III) As measured by the magnitude of the cost functional corresponding to a given set of initial conditions and the rate of convergence of the approximating feedback gains, control law (3.35) is preferable to control law (3.34).

(IV) Using the same criteria as in (III), for the fully discrete schemes, as was the case with the semi-discrete schemes, the spline approximations out-perform the AVE approximations.

5. THEORETICAL CONSIDERATIONS AND FURTHER REMARKS

We turn to a brief discussion of the convergence exhibited by the "gain" operators in $\Pi_N P_N$. Careful study of the numerical examples in section 4 reveals that numerically one has weak convergence of $\Pi_N P_N$, strong (L_2) convergence of the approximating feedback controls $\{\bar{u}_N\}$ of (3.4) and convergence of the performance measures (3.6) and $J(\bar{u}_N;x(0),x_0)$. We have actually proved the weak convergence $\Pi_N P_N \rightharpoonup \Pi$ in certain special cases that include scalar and second order examples such as those considered in section 4. More precisely, if we consider a hereditary system of the form (1.1) and assume (i) (A_0, B_0) is a controllable system and (ii) Range $(B_0) \supset$ Range (A_1), then $\{\Pi_N P_N\}$ is uniformly bounded in $L(Z)$. One can then, under assumptions similar to those invoked by Gibson [14, §6], establish weak convergence of the sequence. For example, if we further assume (iii) the hereditary system is stabilizable, (iv) Q_0, A_0, A_1, B_0 are such that any admissible control drives the state to zero asymptotically, and (v) for N sufficiently large, there exist self-adjoint nonnegative solutions Π_N of the approximate Riccati algebraic equations (3.1), then arguments similar to those in [14] (see Theorem 6.7) can be made to obtain $\Pi_N P_N \rightharpoonup \Pi$.

If we are willing to assume that Gibson's Conjecture 7.1 (essentially, that the approximation scheme when applied preserves uniform asymptotic stability possessed by any original hereditary system) holds for the spline schemes (an assumption that Gibson makes for the "averaging" scheme and one which, based on spectral considerations, we feel confident is valid for both schemes), then we can guarantee existence of solutions Π_N of the approximate Riccati equations (3.1) and furthermore, boundedness of $\{\Pi_N P_N\}$ follows immediately (e.g., see Theorem 7.5 of [14]).

With respect to the question of strong or trace norm convergence of $\{\Pi_N P_N\}$, we note that Gibson [14] obtains such convergence for the averaging scheme. This in turn (see Theorem 6.8 and section 7 of [14]) yields convergence of the payoffs. Fundamental to Gibson's arguments (see Theorems 6.1 and 6.9) is the result $S_N^*(t) \to S^*(t)$ where $\{S(t)\}$ is the solution semigroup for the original hereditary system(2.2) with $u = 0$ and $\{S_N(t)\}$ is the solution semigroup for the approximating system (e.g., (3.2)) with $u = 0$. For our spline schemes we don't believe arguments similar to those of Gibson will suffice to obtain this strong convergence of adjoints; specifically, we don't believe that $A_N^* \to A^*$ in a mode sufficient to yield the required convergence of S_N^*. At this time we honestly don't know whether we have strong convergence of $\{\Pi_N P_N\}$ for the spline schemes. From our numerical results we tend to doubt strong convergence although we <u>do</u> observe the desired convergence of the payoffs and can actually establish this theoretically for our spline schemes. If strong convergence of $\{\Pi_N P_N\}$ is true, we believe a theory somewhat different from that of Gibson's will be required to establish this. In this regard we further note that Kunisch, in his investigation [19] of both the spline and averaging schemes for the finite interval integral quadratic cost control problem for systems with delays, obtains convergence of the payoffs and controls and <u>weak</u> convergence of the associated time dependent Riccati operators in a theoretical treatment that is independent of adjoint convergence considerations. However, even if this theory could be extended to treat the infinite interval regulator problems, it would not appear to yield the stronger convergence results for $\{\Pi_N P_N\}$.

In addition to our numerical findings, there is other evidence that appears to cast doubts on the possiblilty of strong convergence of $\{\Pi_N P_N\}$. Recall that $Z_N \subset D(A)$ for each N and since in the representation

$$\Pi_N P_N = \begin{bmatrix} \Pi_N^{00} & \Pi_N^{01} \\ \\ \Pi_N^{10} & \Pi_N^{11} \end{bmatrix},$$

the first column is "in" D(A), we must have $\Pi_N^{00} = \Pi_N^{10}(0)$. The components of Π do not satisfy such a boundary condition. Indeed (see [14, Theor. 4.4]) col (Π^{00}, Π^{10}) is "in" $D(A^*)$. This suggests that the convergence of $\Pi_N P_N$ to Π cannot occur in a very strong mode.

The theoretical considerations above aside, we believe the evidence is quite substantial in support of our contention that the spline methods offer an attractive means for computing feedback gains in delay system regulator problems. We close with a summary of remarks on the merits of our spline schemes over the averaging scheme (we don't mean to discredit the averaging technique - for many problems it should perform quite admirably - rather we wish to argue that in some examples, the spline schemes can offer significant improvements). We recall from the numerical results of section 4 that (on these examples) the linear spline scheme always is as good as the averaging scheme, in some cases it is better (faster convergence, better approximation at low orders). In some situations the averaging scheme fails numerically to converge, while this never (in our experience) occurs with the spline schemes.

As further evidence of the usefulness of the spline schemes, we offer the recent experiences of Gibson (private communication) and Ito [21] in using the averaging and spline approximation schemes as a basis for computation of closed loop eigenvalues for delay systems (e.g., using the system (3.2), (3.3) to compute eigenvalues that approximate those of the feedback system (2.2), (2.13)). In these efforts, the spline based schemes performed in a far superior manner. We interpret this as another argument in favor of construction of feedback gains via our spline schemes.

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